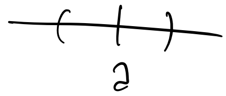
$\sum a_k x^k$ série entière (p. 21)

Taylor

$f : \mathbb{R} \to \mathbb{R}$ $\qquad a \in \mathbb{R}$

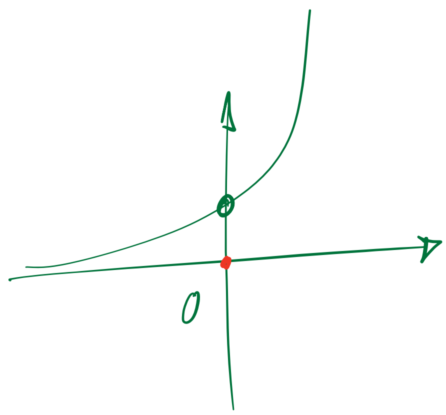


$$f(x) = \sum f^{(k)}(a) \cdot \frac{(x-a)^k}{k!}$$

$$= f(a)\frac{(x-a)^0}{0!} + f'(a)\cdot\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + f^{(3)}(a)\frac{(x-a)^3}{3!} + \dots$$

Développement de Taylor

(fonction $\longrightarrow$ série)



$f(x)\ e^x \qquad a = 0$

$f(0)=1 \quad e^0 = 1$

$(e^x)' = e^x$

$f'(0)=1 \quad e^0 = 1$

$(e^x)'' = e^x$

$f(x) = e^x$

$f^{(k)}(0) = 1$

$$\sum f^{(k)}(0) \cdot \frac{(x-0)^k}{k!}$$

$$= \sum f^{(k)}(0) \cdot \frac{x^k}{k!}$$

$f''(0) \quad e^0 = 1$

$$e^x = f(0) + f'(0)\cdot\frac{(x-0)^1}{1!} + f''(0)\cdot\frac{(x-0)^2}{2!} + f'''(0)\cdot\frac{(x-0)^3}{3!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\Rightarrow \boxed{e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}}$$

$$\frac{\frac{1}{k!}}{\frac{1}{(k+1)!}} = k+1 \to \infty$$

$\Rightarrow r = +\infty$

La serie converge $\forall x \in \mathbb{R}$

$\left(\forall z \in \mathbb{C} \quad e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}\right)$

$$\left(\sum \frac{x^k}{k!}\right)' =$$

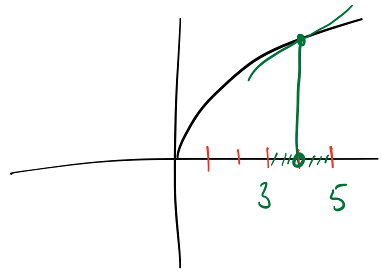
$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)' =$$

$$1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots =$$

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} = \sum \frac{x^k}{k!}$$

$$\Rightarrow (e^x)' = e^x$$

Exemple 1 p. 27



$f(x) = \sqrt{x} \qquad a = 4$

$$f(x) = f(4) + f'(4)\cdot\frac{(x-4)^1}{1!} + f''(4)\cdot\frac{(x-4)^2}{2!} + R_2$$

Ordre 2

$$|R_2(x)| \leq \frac{(x-4)^3}{3!}\cdot \sup_{t\in]3;5[} |f^{(3)}(t)|$$

$$(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}}$$

$f'(4) = \frac{1}{2}\cdot 4^{-\frac{1}{2}} = \frac{1}{4}$

$$(\sqrt{x})'' = (x^{\frac{1}{2}})'' = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} = -\frac{1}{4}x^{-\frac{3}{2}}$$

$f''(4) = -\frac{1}{4}\cdot\frac{1}{8} = -\frac{1}{32}$

$$(\sqrt{x})''' = (x^{\frac{1}{2}})''' = -\frac{1}{4}\cdot\left(-\frac{3}{2}\right)\cdot x^{-\frac{5}{2}} = \frac{3}{8}x^{-\frac{5}{2}}$$

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{32}\,\frac{1}{2!}(x-4)^2 + R_2(x)$$

$$f^{(3)}(x) = \frac{3}{8}x^{-\frac{5}{2}} \qquad \sup_{t\in]3;5[}\left|\frac{3}{8}x^{-\frac{5}{2}}\right| = \sup_{t\in]3;5[}\frac{3}{8(\sqrt{x})^5}$$

$\sqrt{x}$ croissante $\quad (\sqrt{x})^5$ croissante aussi $\Rightarrow \frac{3}{8(\sqrt{x})^5}$ décroissante

$$f^{(3)}(3) = \frac{3}{8(\sqrt{3})^5} \cong 0,024$$

$$|R_2(x)| \leq \frac{|x-4|^3}{3!} \cdot 0,024$$

Erreur max si $x=3$
ou $x=5$

$$\sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + R_2(x)$$

Erreur max : $\frac{1}{3!} \cdot 0,024 = \frac{0,024}{6} = 0,004$

$$\boxed{\text{TE du } 03/02/2025}$$

Exemples 1. à 4. p. 22

Exemples 1. et 2. p. 27

Exercice 16 abcd p. 37

Exercice 19 abc p. 38

Exercices 20, 21 et 22 pp. 38 et 39

$$\sum k^{2k} \cdot x^k$$

$$\ln \left| \frac{k^{2k}}{(k+1)^{2(k+1)}} \right| = \ln \frac{k^{2k}}{(k+1)^{2(k+1)}}$$

$$0 \le \left(\frac{k^k}{(k+1)^{k+1}} \right)^2 = \left(\frac{k^k}{(k+1)^k \cdot (k+1)} \right)^2$$

$$= \left(\frac{k^k}{k^k + A_k} \cdot \frac{1}{k+1} \right)^2$$

$$\le \left(1 \cdot \frac{1}{k+1} \right)^2$$

$$(k+1)^k = k^k + \underbrace{k \cdot k^{k-1} + C_2^k \cdot k^{k-2} + C_3^k \cdot k^{k-3} + \dots}_{A_k}$$

$$(x+y)^n = x^n + C_1^n x^{n-1} y + C_2^n x^{n-2} y^2 + C_3^n x^{n-3} y^3 + \dots + y^n$$

$$k^k + A_k \geq k^k$$

$$\frac{1}{k^k + A_k} \le \frac{1}{k^k} \Rightarrow \frac{k^k}{k^k + A_k} \le \frac{k^k}{k^k} = 1$$

$$\Rightarrow \quad 0 \leq \frac{k^{2k}}{(k+1)^{2(k+1)}} \leq \left(\frac{1}{k+1}\right)^2 \qquad \forall k \geq 1$$

$$\Big\downarrow{\scriptstyle k\to\infty} \qquad \Big\downarrow \qquad \Big\downarrow{\scriptstyle k\to\infty}$$

$$0 \qquad 0 \qquad 0$$

$$\Rightarrow \quad \lim \frac{k^{2k}}{(k+1)^{2(k+1)}} = 0 \qquad \Rightarrow \quad r = 0$$

La série $\sum k^{2k} \cdot x^k$ diverge $\forall x \neq 0$.

Autre méthode :

$$\lim \frac{1}{\sqrt[k]{|a_k|}} = \lim \frac{1}{\sqrt[k]{k^{2k}}} = \lim \frac{1}{\left(\left(k^2\right)^k\right)^{1/k}}$$

$$= \lim \frac{1}{k^2} = 0$$

$$\Rightarrow r = 0$$