Changement de base

$h : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ un endomorphisme

h est donné par sa matrice : $H = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & -2 \\ 2 & 1 & -1 \end{pmatrix}$

$$h(x_1; x_2; x_3) = H \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 - x_3 \\ x_1 + 3x_2 - 2x_3 \\ 2x_1 + x_2 - x_3 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ base canonique

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "H"] & \mathbb{R}^3 \arrow[d, "P^{-1}"] \\
\mathbb{R}^3 \arrow[u, "P"] \arrow[r, "H'"'] & \mathbb{R}^3
\end{tikzcd}$$

$H' = P^{-1} H P$

$\cdot\, u$

$H' \cdot u = P^{-1} \cdot H \cdot P \cdot u$

$h'(u) = P^{-1}(h(p(u)))$

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ autre base

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Calcul de P^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \qquad \begin{array}{l} L_3 \leftarrow L_3 - L_1 \\ L_2 \leftarrow \frac{1}{2} L_2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \qquad \begin{array}{l} L_3 \leftarrow -L_3 \\ L_2 \leftarrow L_2 - \frac{1}{2} L_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \qquad L_1 \leftarrow L_1 - L_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

$$H' = P^{-1} H P$$

$$= \frac{1}{2}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & -2 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 3 & -1 & -1 \\ -1 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}\begin{pmatrix} 0 & 4 & 3 \\ -1 & 5 & 5 \\ 1 & 3 & 2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 4 & 2 \\ 0 & 4 & 4 \\ -2 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow H' = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 2 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

Image et noyau de h donnée par

$$H = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & -2 \\ 2 & 1 & -1 \end{pmatrix}$$

ss-espace vect. de $\mathbb{R}^3$ ($\{0\}$, droite, plan, $\mathbb{R}^3$)

$\{x \in \mathbb{R}^3 \mid h(x) = 0\}$

$\boxed{\ker(h):}$ résoudre $h(x) = 0 \Leftrightarrow H \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

On échelonne H :

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 3 & -2 \\ 2 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} x_1 + 2x_2 - x_3 = 0 \\ x_2 - x_3 = 0 \\ -2x_3 = 0 \end{array} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array} \Rightarrow h(x) = 0 \Leftrightarrow x = 0$$

$\Rightarrow \boxed{\ker(h) = 0}$

$\Rightarrow h$ injectif

$\boxed{\mathrm{Im}(h):}$ h injectif $\Rightarrow h$ surjectif $\Rightarrow \boxed{\mathrm{Im}(h) = \mathbb{R}^3}$

$$G = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix} \qquad g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$\boxed{\ker(g) :}$

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 0 & \qquad x_1 &= -x_2 - 3x_3 = 2k - 3k = -k \\ x_2 + 2x_3 &= 0 & x_2 &= -2k \\ x_3 &= k & x_3 &= k \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

$\boxed{\text{Base du noyau :}}$ $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$, $\ker(g)$ est une droite

$\Rightarrow g$ n'est pas injective $\qquad \dim(\ker(g)) = 1$

Thm du rang : $\underbrace{\dim(\operatorname{Im} g)}^{2} + \underbrace{\dim(\ker g)}^{1} = \dim \mathbb{R}^3 = 3$

$\boxed{Im(g):}$

$$G = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

2 pivots

C_1' C_2'

déjà fait

Colonnes de la matrice éch.

$$\begin{array}{ccc} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, & \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \\ C_1 & C_2 & C_3 \end{array}$$

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$C_3 = C_1 + 2C_2$ lin. dép.

forme paramétrique

$$Im(g) = \left\{ k\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + l\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \;\middle|\; k, l \in \mathbb{R} \right\}$$

$C_1 \neq kC_2$

base de l'image : $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

plan de $\mathbb{R}^3$

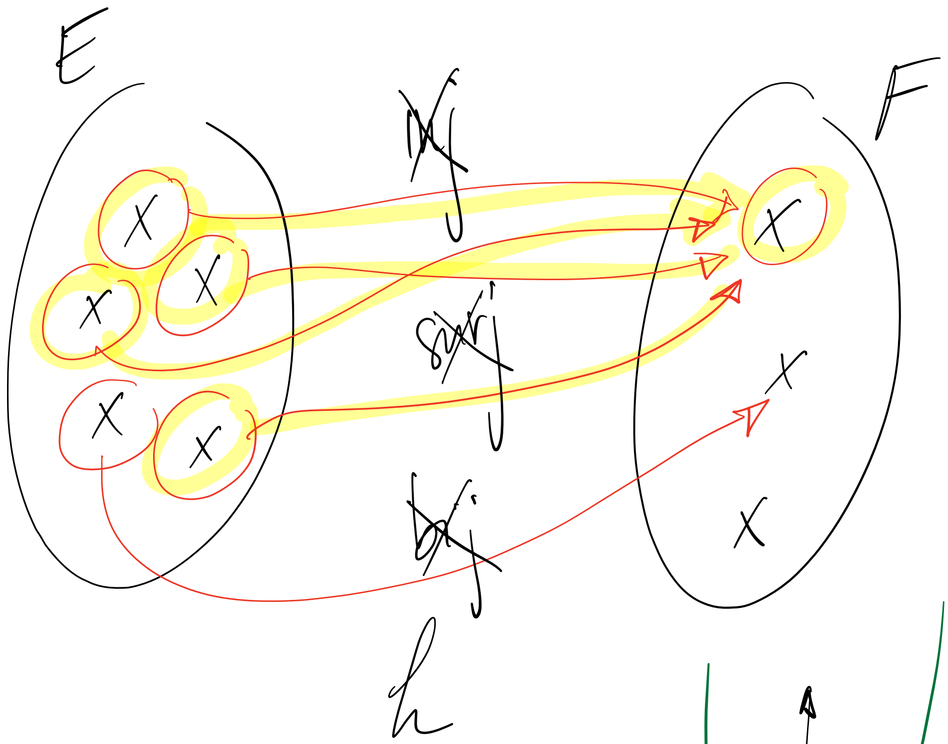
$$\begin{vmatrix} i & 1 & 1 \\ j & 1 & 0 \\ k & 0 & -1 \end{vmatrix} \longrightarrow \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

$\boxed{x - y + z = 0}$ équation du plan

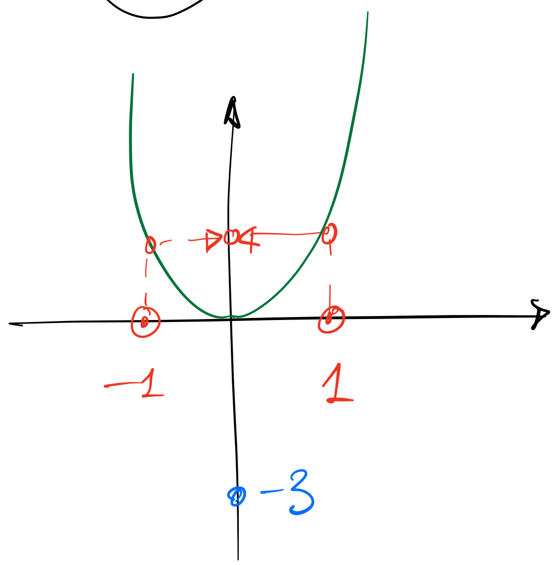
$$
\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "H"] \arrow[d, shift left, "P^{-1}"] & \mathbb{R}^3 \arrow[d, shift left, "P^{-1}"] \\
\mathbb{R}^3 \arrow[r, "H'"'] \arrow[u, shift left, "P"] & \mathbb{R}^3 \arrow[u, shift left, "P"]
\end{tikzcd}
$$

$$H' = P^{-1} \cdot H \cdot P$$

$$H = P\, H' \cdot P^{-1}$$



$$\mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^2$$



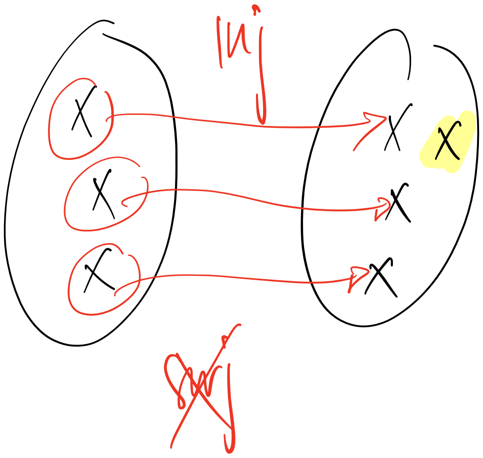
$-1 \mapsto 1 = (-1)^2$

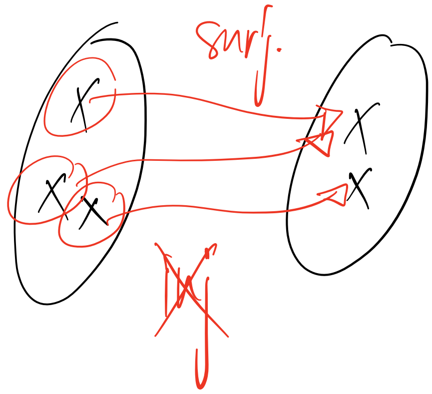
$1 \mapsto 1 = 1^2$

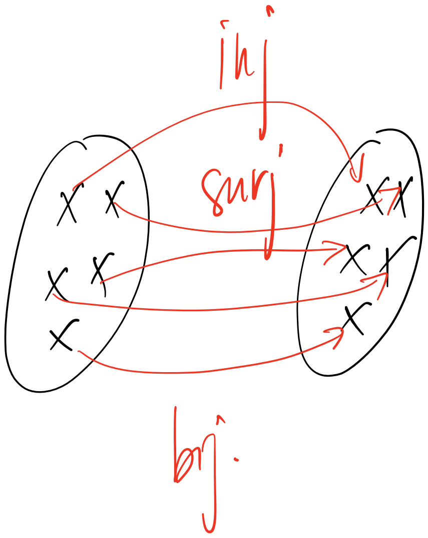
$\cancel{? \mapsto -3}$

$x^2 = -3 \Rightarrow x \in \mathbb{C}$

bij: inj & surj







Cas $h : U \longrightarrow U$

U esp. vect. de dim. finie

h linéaire

$$h(ax+by) = a\,h(x) + b\,h(y)$$

$$h(0) = 0$$

$$\boxed{h \text{ inj.} \Leftrightarrow h \text{ surj.} \Leftrightarrow h \text{ bij.}}$$

$\dim(U) = n \qquad h : U \longrightarrow U$

$\dim(\mathbb{R}^2) = 2$

$\dim(\mathbb{R}^3) = 3$

$\boxed{h \text{ inj.} \Leftrightarrow \ker(h) = 0} \Leftrightarrow \operatorname{Im}(h) = U$

$\boxed{h \text{ bij.} \Leftrightarrow \det(H) \neq 0} \Leftrightarrow \operatorname{Im}(h) = U$

Thm du rang: $\dim(U) = n = \dim(\operatorname{Im} h) + \dim(\ker h)$

$h : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$\dim(\operatorname{Im} h) = 2 \Rightarrow \dim(\ker h) = 1$

$\dim(\operatorname{Im} h) = 1 \Rightarrow \dim(\ker h) = 2$

$\dim(\operatorname{Im} h) = 3 \Rightarrow \dim(\ker h) = 0$