a) $x = u - 1 \Leftrightarrow u = x+1$ $\qquad x=1 / u=2$

$x=3 / u=4$

$$\boxed{dx = du - 0 = du}$$

$$\int_1^3 \frac{x}{\sqrt{x+1}}\,dx = \int_2^4 \frac{u-1}{\sqrt{u-1+1}}\,du$$

$$= \int_2^4 \frac{u-1}{\sqrt{u}}\,du = \int_2^4 \left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right) du$$

$$= \int_2^4 \left(\frac{u^1}{u^{1/2}} - \frac{1}{u^{1/2}}\right) du = \int_2^4 \left(u^{\left(1-\frac{1}{2}\right)} - u^{-\frac{1}{2}}\right) du$$

$$= \int_2^4 u^{\frac{1}{2}}\,du - \int_2^4 u^{-\frac{1}{2}}\,du \qquad \int T^n\,dT = \frac{1}{n+1} T^{n+1}$$

$$= \frac{1}{1+\frac{1}{2}} \cdot u^{1+\frac{1}{2}}\Big|_2^4 - \frac{1}{1-\frac{1}{2}} u^{1-\frac{1}{2}}\Big|_2^4$$

$$= \frac{2}{3}\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right) - 2\left(4^{\frac{1}{2}} - 2^{\frac{1}{2}}\right)$$

$$= \frac{2}{3}\left(8 - \sqrt{8}\right) - 2\left(2 - \sqrt{2}\right)$$

$$= \frac{4}{3} - \frac{2\sqrt{8}}{3} + 2\sqrt{2} = \frac{4}{3} - \frac{4\sqrt{2}}{3} + \frac{6\sqrt{2}}{3}$$

$$= \frac{4}{3} + \frac{2\sqrt{2}}{3} = \frac{2}{3}\left(2 + \sqrt{2}\right)$$

$x \in [1;3]$

b) $x = t^2 - 1 \Leftrightarrow t = \sqrt{x+1}$

$x = 1 \Rightarrow t = \sqrt{2}$

$x = 3 \Rightarrow t = 2$

$$\boxed{dx = 2t\, dt}$$

$$\int_1^3 \frac{x}{\sqrt{x+1}}\, dx = \int_{\sqrt{2}}^{2} \frac{t^2 - 1}{t}\, 2t\, dt$$

$$= \int_{\sqrt{2}}^{2} (2t^2 - 2)\,dt = \left(\frac{2}{3}t^3 - 2t\right)\Big|_{\sqrt{2}}^{2}$$

$$= \left(\frac{2}{3}8 - 4\right) - \left(\frac{2}{3}(\sqrt{2})^3 - 2\sqrt{2}\right)$$

$$= \frac{4}{3} - \frac{4}{3}\sqrt{2} + 2\sqrt{2} = \frac{4}{3} - \frac{4}{3}\sqrt{2} + \frac{6}{3}\sqrt{2}$$

$$= \frac{4}{3} + \frac{2}{3}\sqrt{2} = \frac{2}{3}\left(2 + \sqrt{2}\right)$$

c) $\int x \cdot (x+1)^{-\frac{1}{2}}\,dx$

$(f \cdot g)' = f' g + f \cdot g'$

$(f \cdot g)' = 1 \cdot g \quad x \cdot (x+1)^{-\frac{1}{2}}$

Si $g' = (x+1)^{-\frac{1}{2}}$, $g = \int (x+1)^{-\frac{1}{2}} \, dx$

$$= \int \underbrace{(x+1)}^{-\frac{1}{2}} \cdot 1 \cdot dx = \frac{1}{1-\frac{1}{2}} (x+1)^{1-\frac{1}{2}}$$

dérivée interne

$$= 2 (x+1)^{\frac{1}{2}} = g$$

On a donc : $\left(x \cdot 2 \cdot (x+1)^{\frac{1}{2}} \right)' = 2 \cdot (x+1)^{\frac{1}{2}} + x (x+1)^{-\frac{1}{2}}$

$$\int x (x+1)^{-\frac{1}{2}} dx = x \cdot 2 \cdot (x+1)^{\frac{1}{2}} - \int 2 \cdot \underbrace{(x+1)}_{(\,)'}^{\frac{1}{2}} \cdot 1 \cdot dx$$

$$= 2x (x+1)^{\frac{1}{2}} - 2 \cdot \frac{1}{1+\frac{1}{2}} \cdot (x+1)^{1+\frac{1}{2}}$$

$$= 2x (x+1)^{\frac{1}{2}} - 2 \cdot \frac{2}{3} (x+1)^{\frac{3}{2}} = F(x)$$

$$\Rightarrow \int_1^3 \frac{x}{\sqrt{x+1}}\,dx = F(x)\Big|_1^3 = F(3) - F(1)$$

$$= 2\cdot 3\sqrt{4} - \frac{4}{3}4^{\frac{3}{2}} - \left(2\sqrt{2} - \frac{4}{3}\cdot 2\sqrt{2}\right)$$

$$= 12 - \frac{32}{3} - \sqrt{2}\left(2 - \frac{8}{3}\right)$$

$$= \frac{36-32}{3} - \frac{6-8}{3}\sqrt{2} = \frac{4}{3} + \frac{2}{3}\sqrt{2}$$

$$= \frac{2}{3}\left(2+\sqrt{2}\right)$$