a) $\det \begin{pmatrix} 1-\lambda & 0 & a \\ 1 & -\lambda & -1 \\ -b & -1 & -\lambda \end{pmatrix} = p(\lambda) =$

$= (1-\lambda)\left(\lambda^2 - 1\right) + a\left(-1 - b\lambda\right)$

$\boxed{a = 0}$ $p(\lambda) = 0 \Leftrightarrow \lambda = \pm 1$

Dans ce cas, les valeurs propres de h sont ± 1.

$\boxed{a \neq 0}$ Si $b = -1$, $p(\lambda) = (1-\lambda)^2(1+\lambda) - a(1-\lambda)$

$= (1-\lambda)(1-\lambda^2 - a)$

Pour que $\lambda \in \{-1; 1\}$, il faudrait $a = 0$. ↯

Posons donc $a = 0$.

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ -b & -1 & 0 \end{pmatrix}$$

$$E_1 = \ker \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \\ -b & -1 & -1 \end{pmatrix}$$

$$\begin{cases} x - y - z = 0 \\ -bx - y - z = 0 \end{cases} \qquad \begin{cases} x = y + z \\ (-1-b)\,x = 0 \end{cases}$$

$\boxed{b = -1}$ $\quad E_1 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$\boxed{b \neq -1}$ $\quad E_1 = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$

Vu que $p(\lambda) = (1-\lambda)(\lambda-1)(\lambda+1)$,
il faut que $\dim E_1 = 2$. Si $b \neq 1$
ce n'est pas le cas et h n'est pas
diagonalisable.

Si $b = -1$, on a $H' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

et $P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & a \\ 1 & 0 & -1 \\ -b & -1 & 0 \end{pmatrix} \overset{+b}{\sim} \begin{pmatrix} 1 & 0 & a \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$

$\dim(\mathrm{Im}(h)) = 2 \iff a = -1$

$\iff (1; 1; 0) = k(a; -1; 0)$

On calcule $\ker(h)$ pour $a = -1$

$$\begin{cases} x - z = 0 \\ -6x - y = 0 \end{cases}$$

$\boxed{b = 0}$ $\ker(h) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$\boxed{b \neq 0}$ $\begin{cases} x = z \\ x = -\frac{1}{b} y \end{cases}$ $\begin{cases} x = -\frac{1}{b} y \\ y = y \\ z = -\frac{1}{b} y \end{cases}$

$$\Rightarrow \ker(h) = \left\langle \begin{pmatrix} -1 \\ b \\ -1 \end{pmatrix} \right\rangle$$

On voit donc que $\ker(h) = \langle (-1; b; -1) \rangle \ \forall b$