$f : \mathbb{R}^4 \to \mathbb{R}^2$

a) $f\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix}$ $f\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right) = \begin{pmatrix}1\\2\end{pmatrix}$

$f\left(\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\3\end{pmatrix}$ $f\left(\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\4\end{pmatrix}$

$\Rightarrow F = \begin{pmatrix}1 & 1 & 1 & 1\\1 & 2 & 3 & 4\end{pmatrix}$

b) $F \sim \begin{pmatrix}1 & 1 & 1 & 1\\0 & 1 & 2 & 3\end{pmatrix} \sim \begin{pmatrix}1 & 0 & -1 & -2\\0 & 1 & 2 & 3\end{pmatrix}$

Déterminons une base de $\ker(f)$:

$$\begin{pmatrix}1 & 0 & -1 & -2\\0 & 1 & 2 & 3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$\Leftrightarrow$ $x_1 = x_3 + 2x_4$ $x_3 = x_3$

$x_2 = -2x_3 - 3x_4$ $x_4 = x_4$

$$\Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = k \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + l \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \ker(f) = \left\langle \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

Non, car $\ker(f) \neq \{0\}$

c) Vu que $F \sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$, on a

$$\mathrm{Im}(f) = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; \begin{pmatrix} -1 \\ 2 \end{pmatrix} ; \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = \mathbb{R}^2$$

L'application f est surjective.

d) $u = (x_1 ; x_2 ; 0 ; 0)$

$f(u) = (x_1 + x_2 ; x_1 + 2x_2)$

$$\begin{cases} x_1 + x_2 = y_1 \\ x_1 + 2x_2 = y_2 \end{cases} \iff \begin{cases} x_2 = y_2 - y_1 \\ x_1 = 2y_1 - y_2 \end{cases}$$

$\Rightarrow$ si $u = (2y_1 - y_2 ; y_2 - y_1 ; 0 ; 0)$,

alors $f(u) = (y_1 ; y_2)$