

$$z = x + yi$$

$$\bar{z} = x - yi$$

$$z + \bar{z} = x + yi + x - yi = 2x = 2 \cdot \mathcal{R}(z)$$

$$z - \bar{z} = x + yi - (x - yi) = 2yi = 2 \cdot \mathcal{I}(z) \cdot i$$

$$\begin{aligned} z \cdot \bar{z} &= (x + yi)(x - yi) = x^2 - (yi)^2 \\ &= x^2 - i^2 \cdot y^2 \\ &= x^2 + y^2 \\ &= \mathcal{R}(z)^2 + \mathcal{I}(z)^2 \end{aligned}$$