$$(x+3)(x^2+x+3) = \begin{array}{l} x^3+x^2+3x \\ +3x^2+3x+9 \end{array} = x^3+4x^2+6x+9$$

$$\begin{array}{l|l} x^4 & x^3+4x^2+6x+9 \\ \hline x^4+4x^3+6x^2+9x & x-4 \\ \hline -4x^3-6x^2-9x & \\ -4x^3-16x^2-24x-36 & \\ \hline 10x^2+15x+36 & \end{array}$$

On en déduit :

$$x^4 = (x-4)(x^3+4x^2+6x+9) + 10x^2+15x+36$$

$$\Rightarrow \frac{x^4}{(x+3)(x^2+x+3)} = x-4+\boxed{\frac{10x^2+15x+36}{(x+3)(x^2+x+3)}}$$

À décomposer en éléments simples

$$\frac{10x^2+15x+36}{(x+3)(x^2+x+3)} = \frac{a}{x+3} + \frac{bx+c}{x^2+x+3}$$

On multiplie par $x+3$ et on pose $x=-3$.

$$\frac{10\cdot(-3)^2+15\cdot(-3)+36}{(-3)^2-3+3} = a$$

$$\Leftrightarrow \boxed{a = \frac{81}{9} = 9}$$

On multiplie par x et on passe à la limite.

$$\frac{10x^3+\dots}{x^3+\dots} = \frac{9\cdot x}{x+3} + \frac{bx^2+\dots}{x^2+\dots}$$

$\downarrow x\to\infty \qquad \downarrow$

$$10 = 9 + b \quad \Rightarrow \boxed{b=1}$$

On remplace x par 0.

$$\frac{36}{3\cdot 3} = \frac{9}{3} + \frac{c}{3}$$

$$\Leftrightarrow \quad 4 = 3 + \frac{c}{3} \quad \Leftrightarrow \quad \boxed{c = 3}$$

On peut donc écrire:

$$\frac{x^4}{(x+3)(x^2+x+3)} = x-4 + \frac{9}{x+3} + \frac{x+3}{x^2+x+3}$$

$$\int \frac{x^4}{(x+3)(x^2+x+3)}\,dx = \frac{1}{2}x^2 - 4x + 9\ln|x+3| + \int \frac{x+3}{x^2+x+3}\,dx$$

Reste à calculer la dernière primitive.

$$\frac{x+3}{x^2+x+3} = \frac{x+\frac{1}{2}+\frac{5}{2}}{x^2+x+3}$$

$$= \frac{1}{2}\,\frac{2x+1}{x^2+x+3} + \frac{5}{2}\cdot\frac{1}{x^2+x+3}$$

$$\int \frac{x+3}{x^2+x+3}\,dx = \frac{1}{2}\int \frac{1}{\underbrace{x^2+x+3}}\,\underbrace{(2x+1)}_{()'}\,dx + \frac{5}{2}\int \frac{1}{x^2+x+3}\,dx$$

$$= \frac{1}{2}\ln\left|x^2+x+3\right| + \frac{5}{2}\int \frac{1}{x^2+x+3}\,dx$$

Reste à calculer :

$$\int \frac{1}{x^2+x+3}\,dx =$$

$$\int \frac{1}{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}+3}\,dx =$$

$$\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{11}{4}}\,dx$$

On pose $\left(x+\frac{1}{2}\right)^2 = \frac{11}{4}t^2 \Leftrightarrow x+\frac{1}{2} = \frac{\sqrt{11}}{2}t$

$$dx = \frac{\sqrt{11}}{2}\,dt$$

$$\Rightarrow \int \frac{1}{x^2+x+3}\,dx = \int \frac{1}{\frac{11}{4}t^2+\frac{11}{4}}\,\frac{\sqrt{11}}{2}\,dt$$

$$= \frac{4}{11}\cdot\frac{\sqrt{11}}{2}\int \frac{1}{t^2+1}\,dt$$

$$= \frac{2\sqrt{11}}{11}\cdot \arctan(t)$$

$$= \frac{2\sqrt{11}}{11}\cdot \arctan\left(\frac{2}{\sqrt{11}}\left(x+\frac{1}{2}\right)\right)$$

Finalement,

$$\int \frac{x^4}{(x+3)(x^2+x+3)}\,dx = \frac{1}{2}x^2-4x+9\ln|x+3|+\frac{1}{2}\ln(x^2+x+3)$$
$$+\frac{2\sqrt{11}}{11}\cdot\arctan\left(\frac{2}{\sqrt{11}}\left(x+\frac{1}{2}\right)\right)$$