Racines de nombres complexes

(1) $\sqrt{}$ $\left(\sqrt[4]{} = \sqrt[2]{\sqrt[2]{}}\right)$

$$\sqrt{z} = w \iff z = w^2$$

<u>Thm:</u> $|w^2| = |w|^2$ $\qquad w = a+bi;\ |w| = \sqrt{a^2+b^2}$

<u>preuve:</u> $(a+bi)^2 = a^2 + 2abi + (bi)^2$

$$= a^2 - b^2 + 2abi$$

$$|w^2| = |(a+bi)^2| = \sqrt{(a^2-b^2)^2 + (2ab)^2}$$

$$= \sqrt{a^4 - 2a^2b^2 + b^4 + 4a^2b^2}$$

$$= \sqrt{a^4 + 2a^2b^2 + b^4}$$

$$= \sqrt{(a^2+b^2)^2} = a^2+b^2 = |w|^2$$

$$= \left(\sqrt{a^2+b^2}\right)^2$$

CQFD

$w^2 = z \Rightarrow |w|^2 = |z| \Rightarrow a^2+b^2 = |z|$

$w = a+bi$

connu

On a également $a^2-b^2+2abi = z$

Soit $z = 3+4i$

$a^2-b^2+2abi = 3+4i \Leftrightarrow \begin{cases} a^2-b^2=3 \\ 2ab=4 \end{cases}$

De plus, $a^2+b^2 = |z| = \sqrt{9+16} = 5$

A résoudre: $\begin{cases} a^2-b^2=3 \\ a^2+b^2=5 \\ 2ab=4 \end{cases}$ $2a^2=8 \,/\, a^2=4 \,/\, a=\pm 2$

$b = \pm 1$

$\Rightarrow w = \pm(2+i)$

2. $\sqrt[k]{} = ()^{1/k}$

$z = a+bi$

$z = r(\cos\theta + \sin\theta i) = [r; \theta]$

__Remarque:__ $z_1 = [r_1 ; \theta_1]$ $z_2 = [r_2 ; \theta_2]$

$$z_1 z_2 = [r_1 \cdot r_2 ; \theta_1 + \theta_2]$$

Car $z_1 z_2 = r_1 (\cos\theta_1 + \sin\theta_1 \cdot i) \, r_2 (\cos\theta_2 + \sin\theta_2 \cdot i)$

$$= r_1 r_2 \cdot \Big(\underbrace{\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2}_{\cos(\theta_1+\theta_2)} + \underbrace{(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)}_{\sin(\theta_1+\theta_2)} i \Big)$$

$$= r_1 r_2 \big(\cos(\theta_1+\theta_2) + \sin(\theta_1+\theta_2) \cdot i \big)$$

$$= [r_1 r_2 ; \theta_1 + \theta_2]$$

Formule de Moivre:

$$z^n = [r ; \theta]^n = [r^n ; n\theta]$$

Par réc. sur n. Si $n=1$, c'est clair. Si $n=2$, voir ci-dessus

$\boxed{P(n) \Rightarrow P(n+1)}$

$$z^{n+1} = z^n \cdot z \overset{\text{Hyp. de réc.}}{=} [r^n ; n\theta] \cdot [r ; \theta]$$

$$= r^n(\cos n\theta + \sin n\theta \cdot i) \cdot r \cdot (\cos\theta + \sin\theta\, i)$$

$$= r^{n+1}(\cos n\theta \cos\theta - \sin n\theta \sin\theta + (\cos n\theta \sin\theta + \sin n\theta \cos\theta) i)$$

$$= r^{n+1}(\cos(n\theta + \theta) + \sin(n\theta + \theta) \cdot i)$$

$$= r^{n+1}(\cos(n+1)\theta + \sin(n+1)\theta \cdot i) \quad \text{CQFD}$$

À calculer:

$$\sqrt[k]{z} = [r; \theta] \implies z = [r;\theta]^k$$

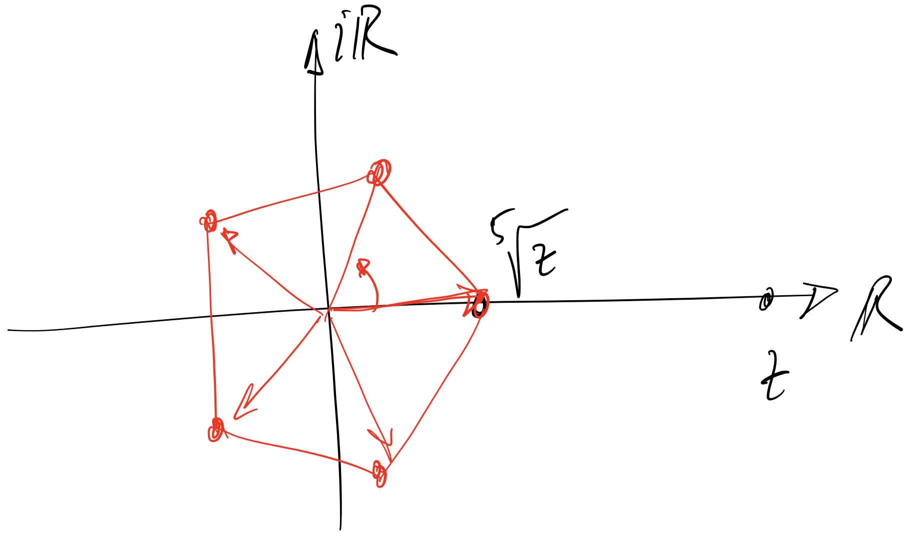
$$\left.\begin{aligned} z &= [r^k; k\cdot\theta] \\ z &= [s; \varphi] \end{aligned}\right\} \Rightarrow s = r^k \Rightarrow r = \sqrt[k]{s}$$

$r, s \in [0; +\infty[$

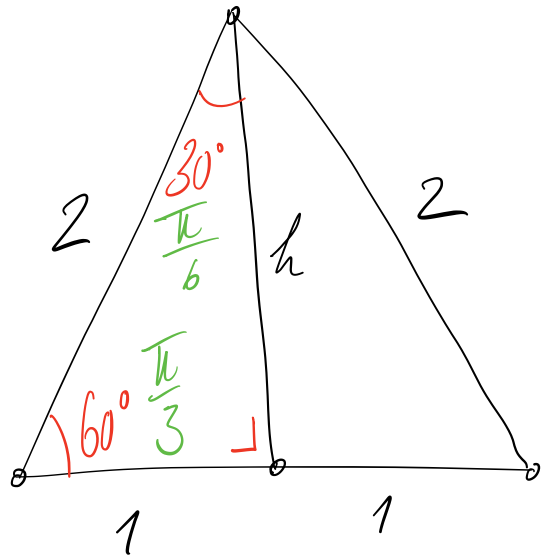
$$k \cdot \theta = \varphi + n \cdot 2\pi$$

$$\theta = \frac{\varphi}{k} + \frac{n \cdot 2\pi}{k} \qquad n \in \mathbb{Z}_k$$

k valeurs possibles: $0, \ldots, k-1$


$i\mathbb{R}$
$\sqrt[5]{z}$
$\mathbb{R}$
z

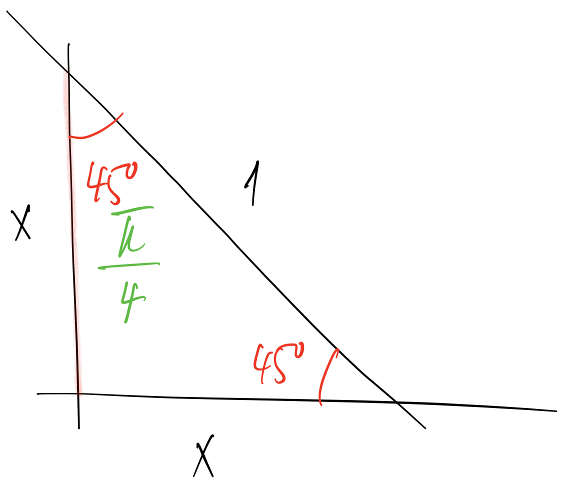
Valeurs exactes



$h^2 = 2^2 - 1^2$

$h = \sqrt{3}$

$\Rightarrow \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ / $\cos\frac{\pi}{3} = \frac{1}{2}$ / ...



$2x^2 = 1$ / $x^2 = \frac{1}{2}$ / $x = \frac{\sqrt{2}}{2}$

$\tan\frac{\pi}{4} = 1$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}/2}{1} = \frac{\sqrt{2}}{2}$

...

$$\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$